

PSEUDO-RIEMANNIAN $G_{2(2)}$ -MANIFOLDS WITH DIMENSION AT MOST 21

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ABSTRACT. Let $G_{2(2)}$ be the non-compact connected simple Lie group of type G_2 over \mathbb{R} , and let M be a connected analytic complete pseudo-Riemannian manifold that admits an isometric $G_{2(2)}$ -action with a dense orbit. For the case $\dim(M) \leq 21$, we provide a full description of the manifold M , its geometry and its $G_{2(2)}$ -action. The latter are always given in terms of a Lie group geometry related to $G_{2(2)}$, and in one case M is essentially the quotient of $\mathrm{SO}_0(3, 4)$ by a lattice.

1. INTRODUCTION

A fundamental problem in both geometry and dynamics is to understand the actions of a connected simple Lie group G on manifolds. This is particularly interesting when one of such G -actions preserves a geometric structure on a manifold M . A basic example to consider is the left G -action on the manifold H/Γ where H is a semisimple Lie group, Γ is a lattice of H and the action is given by a non-trivial homomorphism $G \rightarrow H$. There are two well known properties for such an example. In the first place, the G -action on H/Γ is ergodic and so has a dense orbit. And secondly, the Killing form of the Lie algebra of H induces a bi-invariant pseudo-Riemannian metric on H that descends to a metric on H/Γ preserved by the G -action.

It has been conjectured that every finite volume preserving ergodic G -action on a manifold is essentially one of the examples H/Γ just described (see [16]). In this direction, Zimmer's program proposes to study ergodic G -actions to understand their rigid properties. Many efforts on this line of research have shown very useful to consider actions that preserve a geometric structure. In particular, this has lead to the development of a set of tools widely known as Gromov-Zimmer's machinery (see for example [2, 15, 11]).

In this work we consider the case $G = G_{2(2)}$, the connected non-compact exceptional Lie group of type G_2 over \mathbb{R} , and an isometric $G_{2(2)}$ -action on a finite volume pseudo-Riemannian manifold M . Following Zimmer's program, the final goal is to prove that M is closely related to the group $G_{2(2)}$ itself, from the viewpoint of all the structures involved. We achieve this objective for the case $\dim(M) \leq 21$. We note that 21 is the dimension of the Lie group $\mathrm{SO}(3, 4)$ and that there is a homomorphism $G_{2(2)} \rightarrow \mathrm{SO}(3, 4)$ that realizes the irreducible non-trivial representation of $G_{2(2)}$ with lowest dimension (see Section 1 for details). The following result proves that M is always related to $G_{2(2)}$ and in one case that it is essentially given

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by a non-trivial homomorphism $G_{2(2)} \rightarrow \mathrm{SO}(3, 4)$. In a sense, we thus provide a geometric/dynamic characterization of the homomorphism $G_{2(2)} \rightarrow \mathrm{SO}(3, 4)$ that defines the irreducible 7-dimensional representation of $G_{2(2)}$.

Theorem 1.1 (Manifold and action type). *Let (M, h) be a connected analytic complete pseudo-Riemannian manifold with finite volume that admits an isometric $G_{2(2)}$ -action with a dense orbit. If $\dim M \leq 21$, then there exists a finite covering map $\pi : \widehat{M} \rightarrow M$ so that \widehat{M} satisfies one of the following properties.*

- (1) *There exist a connected pseudo-Riemannian manifold N and a discrete subgroup $\Gamma \subset G_{2(2)} \times \mathrm{Iso}(N)$ such that*

$$\widehat{M} = (G_{2(2)} \times N) / \Gamma.$$

Furthermore, the $G_{2(2)}$ -action on M lifted to \widehat{M} is precisely the left $G_{2(2)}$ -action induced from the action on the first factor of $G_{2(2)} \times N$.

- (2) *There exist a lattice $\Gamma \subset \widetilde{\mathrm{SO}}_0(3, 4)$ so that*

$$\widehat{M} = \widetilde{\mathrm{SO}}_0(3, 4) / \Gamma.$$

Furthermore, the $G_{2(2)}$ -action on M lifted to \widehat{M} is precisely the left $G_{2(2)}$ -action induced from a non-trivial homomorphism $G_{2(2)} \rightarrow \widetilde{\mathrm{SO}}_0(3, 4)$ and the left translation action on $\widetilde{\mathrm{SO}}_0(3, 4)$.

The next result proves that the pseudo-Riemannian metric on M can also be related to natural metrics.

Theorem 1.2 (Metric type). *With the hypotheses and notation of Theorem 1.1, one of the following holds according to the cases of such theorem.*

- (1) *For $\widehat{M} = (G_{2(2)} \times N) / \Gamma$, the covering map $\pi : (\widehat{M}, \widehat{h}) \rightarrow (M, h)$ is locally isometric for the metric \widehat{h} on \widehat{M} induced from the product metric on $G_{2(2)} \times N$ where $G_{2(2)}$ carries a bi-invariant metric.*
- (2) *For $\widehat{M} = \widetilde{\mathrm{SO}}_0(3, 4) / \Gamma$, there is new $G_{2(2)}$ -invariant metric \overline{h} on M so that the covering map $\pi : (\widehat{M}, \widehat{h}) \rightarrow (M, \overline{h})$ is locally isometric for the metric \widehat{h} on \widehat{M} induced from the bi-invariant metric on $\widetilde{\mathrm{SO}}_0(3, 4)$ given by the Killing form of $\mathfrak{so}(3, 4)$.*

It is well known that the group $\widetilde{\mathrm{SO}}_0(3, 4)$ is weakly irreducible for the bi-invariant metric defined by the Killing form of its Lie algebra. We recall that a pseudo-Riemannian manifold is weakly irreducible if its not locally isomorphic to a product of pseudo-Riemannian manifolds. This property can be used to distinguish between the two cases of the theorems above.

As for the organization of the work, in Section 2 we present some basic facts on $G_{2(2)}$, its Lie algebra $\mathfrak{g}_{2(2)}$ and their representations. Section 3 applies the Gromov-Zimmer's machinery to describe the centralizer \mathcal{H} of the $G_{2(2)}$ -action on the universal covering space \widetilde{M} in the Lie algebra of Killing fields. Finally, Section 4 provide the proofs of the results stated in this Introduction.

2. PRELIMINARIES ON $G_{2(2)}$

We introduce the exceptional Lie group $G_{2(2)}$ and recall some properties that we will use in this work. This includes some properties of $\mathfrak{g}_{2(2)}$, the Lie algebra of $G_{2(2)}$. We refer to [5, 12] for further details.

We define $G_{2(2)}$ as the connected group of automorphisms of the split Cayley algebra \mathcal{C} over \mathbb{R} . We recall that \mathcal{C} is a composition algebra whose norm is a split quadratic form. In other words, the norm of \mathcal{C} is a quadratic form whose associated bilinear form has signature $(4, 4)$. The group $G_{2(2)}$ preserves the unit e of \mathcal{C} and so it preserves the orthogonal complement e^\perp which is precisely the space of pure imaginary elements of \mathcal{C} : the set of $a \in \mathcal{C}$ such that $\bar{a} = -a$. The bilinear form of \mathcal{C} restricted to e^\perp has signature $(3, 4)$ and so we will denote $e^\perp = \mathbb{R}^{3,4}$. This yields a faithful representation $G_{2(2)} \rightarrow \mathrm{SO}(3, 4)$, that we will call the linear realization of $G_{2(2)}$. Correspondingly, there is a Lie algebra representation $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$, that exhibits $\mathfrak{g}_{2(2)}$ as the Lie algebra of derivations of \mathcal{C} restricted to e^\perp . We will call this representation the linear realization of $\mathfrak{g}_{2(2)}$.

Proposition 2.1. *The Lie algebra $\mathfrak{g}_{2(2)}$ is the split exceptional Lie algebra of type G_2 over \mathbb{R} . The linear realization $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$ turns the space $\mathbb{R}^{3,4}$ into an irreducible $\mathfrak{g}_{2(2)}$ -module. Furthermore, $\mathbb{R}^{3,4}$ is the $\mathfrak{g}_{2(2)}$ -module corresponding to the first fundamental weight of $\mathfrak{g}_{2(2)}$ and every other irreducible $\mathfrak{g}_{2(2)}$ -module has dimension at least 14.*

Proof. The first claim is well known (see [5, 12]).

Next, we recall that the irreducible representations of the split form \mathfrak{g} of a simple complex Lie algebra $\mathfrak{g}^\mathbb{C}$ are all real forms of the corresponding irreducible representations of $\mathfrak{g}^\mathbb{C}$ (see [9]). On the other hand, the irreducible representation of $\mathfrak{g}_2^\mathbb{C}$ corresponding to the first fundamental weight has dimension 7, and all other irreducible non-trivial representations have dimension at least 14 (see [4]). Hence, the irreducible representation of $\mathfrak{g}_{2(2)}$ associated to the first fundamental weight has (real) dimension 7. Since the linear realization homomorphism $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$ is non-trivial, it defines such irreducible representation.

Finally, we recall that Weyl's dimension formula implies that the irreducible representations of $\mathfrak{g}_2^\mathbb{C}$ corresponding to the first and second fundamental weights have dimensions 7 and 14, and that every other irreducible representations has dimension strictly larger (see [4]). Hence, the last claim follows from the above remarks on split forms. \square

In the rest of this work, $\mathbb{R}^{3,4}$ will be considered as a $\mathfrak{g}_{2(2)}$ -module with the structure given by Proposition 2.1.

The following result establishes the uniqueness of the $\mathfrak{g}_{2(2)}$ -invariant scalar product on $\mathbb{R}^{3,4}$.

Proposition 2.2. *The $\mathfrak{g}_{2(2)}$ -module $\mathbb{R}^{3,4}$ carries a unique, up to a constant multiple, scalar product invariant under $\mathfrak{g}_{2(2)}$. In particular, any such scalar product has signature either $(3, 4)$ or $(4, 3)$.*

Proof. The existence of the scalar product is clear from the construction of the $\mathfrak{g}_{2(2)}$ -module $\mathbb{R}^{3,4}$ in terms the composition algebra \mathcal{C} .

Recall that there is a natural isomorphism of vector spaces between the space of $\mathfrak{g}_{2(2)}$ -invariant bilinear forms on $\mathbb{R}^{3,4}$ and the real algebra $\text{End}_{\mathfrak{g}_{2(2)}}(\mathbb{R}^{3,4})$ of homomorphisms of $\mathfrak{g}_{2(2)}$ -modules of $\mathbb{R}^{3,4}$. Hence, it is enough to prove that $\text{End}_{\mathfrak{g}_{2(2)}}(\mathbb{R}^{3,4})$ is 1-dimensional.

By Schur's Lemma and the irreducibility of $\mathbb{R}^{3,4}$, the algebra $\text{End}_{\mathfrak{g}_{2(2)}}(\mathbb{R}^{3,4})$ is a division algebra over \mathbb{R} , and so it is isomorphic to either \mathbb{R} , \mathbb{C} or the quaternion numbers. If $\text{End}_{\mathfrak{g}_{2(2)}}(\mathbb{R}^{3,4})$ is not 1-dimensional, then there is a $\mathfrak{g}_{2(2)}$ -invariant complex structure on $\mathbb{R}^{3,4}$, which is absurd since this space is odd-dimensional. \square

We recall the following elementary property.

Lemma 2.3. *Let E be a finite dimensional vector space with scalar product $\langle \cdot, \cdot \rangle$. Then, the assignment*

$$u \wedge v \mapsto \langle \cdot, u \rangle v - \langle \cdot, v \rangle u,$$

defines an isomorphism $\varphi : \wedge^2 E \rightarrow \mathfrak{so}(E)$ of $\mathfrak{so}(E)$ -modules. In particular, φ yields an isomorphism of \mathfrak{g} -modules for every Lie subalgebra \mathfrak{g} of $\mathfrak{so}(E)$.

As a consequence $\wedge^2 \mathbb{R}^{3,4} \simeq \mathfrak{so}(3, 4)$ as $\mathfrak{g}_{2(2)}$ -modules for the structures defined by the linear realization of $\mathfrak{g}_{2(2)}$. The next proposition describes some useful properties of these $\mathfrak{g}_{2(2)}$ -modules.

Remark 2.4. If \mathfrak{h} is a Lie algebra, then the Jacobi identity implies that the linear map

$$\begin{aligned} \wedge^2 \mathfrak{h} &\rightarrow \mathfrak{h} \\ X \wedge Y &\mapsto [X, Y], \end{aligned}$$

is a homomorphism of \mathfrak{h} -modules. In particular, if \mathfrak{h}_1 is a Lie subalgebra of \mathfrak{h} and V_1, V_2 are \mathfrak{h}_1 -submodules of \mathfrak{h} (for the \mathfrak{h}_1 -module structure defined by the Lie brackets), then $[V_1, V_2]$ is an \mathfrak{h}_1 -module (again, by the Jacobi identity) whose irreducible \mathfrak{h}_1 -submodules must be among those that appear in $V_1 \otimes V_2$. Similarly, there is a corresponding remark for $[V_1, V_1]$ and $\wedge^2 V_1$. We will use these facts in the rest of this work.

Proposition 2.5. *The following isomorphism of $\mathfrak{g}_{2(2)}$ -modules holds*

$$\wedge^2 \mathbb{R}^{3,4} \simeq \mathfrak{so}(3, 4) \simeq \mathbb{R}^{3,4} \oplus \mathfrak{g}_{2(2)},$$

where $\mathfrak{g}_{2(2)}$ is the $\mathfrak{g}_{2(2)}$ -module given by the adjoint representation. If we let V denote the $\mathfrak{g}_{2(2)}$ -submodule of $\mathfrak{so}(3, 4)$ isomorphic to $\mathbb{R}^{3,4}$, then

$$[V, V] = \mathfrak{so}(3, 4),$$

with respect to the Lie brackets of $\mathfrak{so}(3, 4)$.

Proof. Since $\mathfrak{g}_{2(2)}$ is $\mathfrak{g}_{2(2)}$ -submodule of $\mathfrak{so}(3, 4)$ (for the structure mentioned above) and since $\mathfrak{g}_{2(2)}$ is simple, there is a $\mathfrak{g}_{2(2)}$ -submodule V of $\mathfrak{so}(3, 4)$ such that

$$\mathfrak{so}(3, 4) = \mathfrak{g}_{2(2)} \oplus V.$$

In particular, V has dimension 7. By Proposition 2.1, either V is a direct sum of trivial 1-dimensional modules or $V \simeq \mathbb{R}^{3,4}$ as $\mathfrak{g}_{2(2)}$ -modules. If the former occurs, then Remark 2.4 implies that $[V, V]$ is a sum of 1-dimensional $\mathfrak{g}_{2(2)}$ -modules as well and so $[V, V] \subset V$. This implies that V is a proper ideal of $\mathfrak{so}(3, 4)$, which is absurd. This proves the first claim.

On the other hand, by Remark 2.4 the space $[V, V]$ is either 0 or a sum of the irreducible $\mathfrak{g}_{2(2)}$ -modules that appear in $\mathfrak{so}(3, 4)$. We have already ruled out that $[V, V] \subset V$. If $[V, V] \subset \mathfrak{g}_{2(2)}$, then $(\mathfrak{so}(3, 4), \mathfrak{g}_{2(2)})$ is a symmetric pair. But an inspection of Table II from [1] shows that no such symmetric pair exists. Therefore, the only possibility left is to have $[V, V] = \mathfrak{so}(3, 4)$. \square

Remark 2.6. Note that $\mathfrak{g}_{2(2)}$, as a module over itself, is the irreducible representation of $\mathfrak{g}_{2(2)}$ corresponding to the second fundamental weight. Hence, Proposition 2.5 says that the $\mathfrak{g}_{2(2)}$ -module $\wedge^2 \mathbb{R}^{3,4} \simeq \mathfrak{so}(3, 4)$ is the sum of the irreducible representations corresponding to the fundamental weights.

3. CENTRALIZER OF THE ISOMETRIC $G_{2(2)}$ -ACTION

In this section we specialize some known results for actions of non-compact simple Lie groups to the our case of $G_{2(2)}$ -actions. Our main references are [6] and [7].

We will assume the hypotheses of Theorem 1.1 through out this section. Under such conditions, it is well known that the $G_{2(2)}$ -action on M is everywhere locally free (see [13]). Hence, the set of orbits defines a foliation \mathcal{O} on M , whose tangent bundle will be denoted by $T\mathcal{O}$. In particular, the map $M \times \mathfrak{g}_{2(2)} \rightarrow T\mathcal{O}$ given by the assignment $(x, X) \mapsto X_x^*$ is an isomorphism of bundles. We recall that for $X \in \mathfrak{g}_{2(2)}$ we denote by X^* the vector field on M whose local flow is $\exp(tX)$. Also, we will denote by $T\mathcal{O}^\perp$ the bundle whose fibers are the subspaces orthogonal to the fibers of $T\mathcal{O}$. Then, the condition $\dim(M) \leq 21$ ensures that both $T\mathcal{O}$ and $T\mathcal{O}^\perp$ are non-degenerate and so that $TM = T\mathcal{O} \oplus T\mathcal{O}^\perp$ (see Lemma 1.4 from [6]). In what follows, we will use the same symbols \mathcal{O} , $T\mathcal{O}$ and $T\mathcal{O}^\perp$ for the corresponding objects on \widetilde{M} .

The following result is fundamental for our work. See Proposition 2.3 from [11] for a proof for arbitrary non-compact simple Lie groups (see also [2, 15]). For a pseudo-Riemannian manifold N we denote by $\text{Kill}(N)$ the Lie algebra of globally defined Killing vector fields on N . Also, we will denote by $\text{Kill}_0(N, x)$ the Lie subalgebra of $\text{Kill}(N)$ consisting of those vector fields that vanish at x .

Proposition 3.1. *For M as above, there is a dense conull subset $A \subset \widetilde{M}$ such that for every $x \in A$ the following properties are satisfied.*

- (1) *There is a homomorphism $\rho_x : \mathfrak{g}_{2(2)} \rightarrow \text{Kill}(\widetilde{M})$ which is an isomorphism onto its image $\rho_x(\mathfrak{g}_{2(2)})$.*
- (2) *Every element of $\rho_x(\mathfrak{g}_{2(2)})$ vanishes at x : $\rho_x(\mathfrak{g}_{2(2)}) \subset \text{Kill}_0(\widetilde{M}, x)$.*
- (3) *For every $X, Y \in \mathfrak{g}_{2(2)}$ we have:*

$$[\rho_x(X), Y^*] = [X, Y]^* = -[X^*, Y^*].$$

In particular, the elements in $\rho_x(\mathfrak{g}_{2(2)})$ and their corresponding local flows preserve both \mathcal{O} and $T\mathcal{O}^\perp$.

The following local homogeneity result is well known and it is a particular case of Gromov's open dense orbit theorem. For its proof for general actions of non-compact simple Lie groups we refer to [2] and [15].

Proposition 3.2. *For M satisfying the above conditions, there is an open dense conull subset $U \subset \widetilde{M}$ such that for every $x \in U$ the evaluation map $ev_x : \mathcal{H} \rightarrow T_x \widetilde{M}$ given by $Z \mapsto Z_x$ is surjective.*

For A as in Proposition 3.1, let $x \in A$ be given and consider the map

$$\begin{aligned}\widehat{\rho}_x : \mathfrak{g}_{2(2)} &\rightarrow \text{Kill}(\widetilde{M}) \\ \widehat{\rho}_x(X) &= \rho_x(X) + X^*.\end{aligned}$$

Then, Proposition 3.1(3) implies that $\widehat{\rho}_x$ is an injective homomorphism of Lie algebras. We will denote its image by $\mathcal{G}(x)$, which is thus a Lie subalgebra of \mathcal{H} isomorphic to $\mathfrak{g}_{2(2)}$. In particular, the Lie brackets induce a $\mathfrak{g}_{2(2)}$ -module structure on \mathcal{H} . Furthermore, through the isomorphism $\widehat{\rho}_x$ between $\mathcal{G}(x)$ and $\mathfrak{g}_{2(2)}$ every $\mathcal{G}(x)$ -module can be considered as a $\mathfrak{g}_{2(2)}$ -module.

Proposition 3.2 allows us to define a $\mathcal{G}(x)$ -module structure on $T_x\widetilde{M}$ through the following construction.

Let A and U be as in Propositions 3.1 and 3.2, respectively. Fix some point $x \in A \cap U$. We consider the map $\lambda_x : \mathcal{G}(x) \rightarrow \mathfrak{so}(T_x\widetilde{M})$ given by

$$\lambda_x(Z)(v) = [Z, V]_x,$$

where $V \in \mathcal{H}$ is such that $V_x = v$. It is easy to see that this is a well defined homomorphism of Lie algebras. Furthermore, it is also known that the evaluation map $ev_x : \mathcal{H} \rightarrow T_x\widetilde{M}$ is a homomorphism of $\mathcal{G}(x)$ -modules that satisfies $ev_x(\mathcal{G}(x)) = T_x\mathcal{O}$. In particular, $T_x\mathcal{O}$ is a $\mathcal{G}(x)$ -module isomorphic to the $\mathfrak{g}_{2(2)}$ -module $\mathfrak{g}_{2(2)}$. As a consequence the subspace $T_x\mathcal{O}^\perp$ is a $\mathcal{G}(x)$ -submodule of $T_x\widetilde{M}$.

For $x \in A \cap U$, in the rest of this work we consider \mathcal{H} and $T_x\widetilde{M}$ endowed with the $\mathcal{G}(x)$ -module structures defined above.

On the other hand, we denote by $\mathcal{H}_0(x) = \ker(ev_x)$ which, by the previous remarks, is a $\mathcal{G}(x)$ -submodule of \mathcal{H} . Also, it is clear that $\mathcal{H}_0(x)$ is a Lie subalgebra of \mathcal{H} as well. In particular, $\mathcal{G}(x) + \mathcal{H}_0(x)$ is a Lie subalgebra of \mathcal{H} that contains $\mathcal{H}_0(x)$ as an ideal. Hence, \mathcal{H} can be considered as a module over $\mathcal{G}(x) + \mathcal{H}_0(x)$ through the Lie brackets.

By Proposition 3.5 from [7] we can extend λ_x from $\mathcal{G}(x)$ to the map

$$\begin{aligned}\lambda_x : \mathcal{G}(x) + \mathcal{H}_0(x) &\rightarrow \mathfrak{so}(T_x\widetilde{M}) \\ \lambda_x(Z)(v) &= [Z, V]_x,\end{aligned}$$

where for a given $v \in T_x\widetilde{M}$ we choose $V \in \mathcal{H}$ such that $V_x = v$. As before, it is proved that λ_x is a well defined homomorphism of Lie algebras, thus defining a $\mathcal{G}(x) + \mathcal{H}_0(x)$ -module structure on $T_x\widetilde{M}$ for which both $T_x\mathcal{O}$ and $T_x\mathcal{O}^\perp$ are submodules. Furthermore, the evaluation map $ev_x : \mathcal{H} \rightarrow T_x\widetilde{M}$ is a homomorphism of $\mathcal{G}(x) + \mathcal{H}_0(x)$ -modules. In particular, we have a representation

$$\begin{aligned}\lambda_x^\perp : \mathcal{G}(x) + \mathcal{H}_0(x) &\rightarrow \mathfrak{so}(T_x\mathcal{O}^\perp) \\ \lambda_x^\perp(Z) &= \lambda_x(Z)|_{T_x\mathcal{O}^\perp}.\end{aligned}$$

Furthermore, by Proposition 3.6 from [7], the restriction $\lambda_x^\perp : \mathcal{H}_0(x) \rightarrow \mathfrak{so}(T_x\mathcal{O}^\perp)$ is injective and its image is both a Lie subalgebra and a $\mathcal{G}(x)$ -submodule of $\mathfrak{so}(T_x\mathcal{O}^\perp)$.

The fact that $\mathcal{G}(x) \simeq \mathfrak{g}_{2(2)}$ as a Lie algebra allows us to obtain the following decomposition of the centralizer \mathcal{H} .

Proposition 3.3. *Let A and U be as in Propositions 3.1 and 3.2, respectively. For a fixed point $x \in A \cap U$ there exists a $\mathcal{G}(x)$ -submodule $\mathcal{V}(x)$ of \mathcal{H} such that*

$$\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{H}_0(x) \oplus \mathcal{V}(x), \quad T_x\mathcal{O}^\perp = ev_x(\mathcal{V}(x)).$$

Next we consider the analytic map

$$\omega : T\widetilde{M} \rightarrow \mathfrak{g}_{2(2)}$$

given by the orthogonal projection onto $T\mathcal{O}$ followed by the fiberwise isomorphism $T\mathcal{O} \rightarrow \mathfrak{g}_{2(2)}$ described at the beginning of this section. Let us also consider the analytic $\mathfrak{g}_{2(2)}$ -valued 2-form Ω defined by

$$\Omega_x = d\omega_x|_{\wedge^2 T_x \mathcal{O}^\perp},$$

for every $x \in \widetilde{M}$. If X, Y are smooth sections of $T\mathcal{O}^\perp$, then $\omega(X) = \omega(Y) = 0$ and so we have

$$\Omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]) = -\omega([X, Y]),$$

which implies the following result (see [2, 11]).

Lemma 3.4. *For G and M as above, assume that $T\widetilde{M} = T\mathcal{O} \oplus T\mathcal{O}^\perp$. Then, $T\mathcal{O}^\perp$ is integrable if and only if $\Omega \equiv 0$.*

By Lemma 2.3, we obtain from the map $\Omega_x : \wedge^2 T_x \mathcal{O}^\perp \rightarrow \mathfrak{g}_{2(2)}$ a corresponding map $\mathfrak{so}(T_x \mathcal{O}^\perp) \rightarrow \mathfrak{g}_{2(2)}$ given by $\Omega_x \circ \varphi_x^{-1}$, where $\varphi_x : \wedge^2 T_x \mathcal{O}^\perp \rightarrow \mathfrak{so}(T_x \mathcal{O}^\perp)$ is the isomorphism defined by Lemma 2.3. This does not change the $\mathfrak{so}(T_x \mathcal{O}^\perp)$ -module structure on the domain. Hence, we will denote with the same symbol Ω_x the linear map given by the 2-form Ω when considered as a map $\mathfrak{so}(T_x \mathcal{O}^\perp) \rightarrow \mathfrak{g}$.

It turns out that the forms ω_x and Ω_x satisfy special intertwining properties with respect to the module structure over $\mathcal{G}(x) + \mathcal{H}_0(x)$. These are stated and proved in Proposition 3.10 from [7] for general non-compact simple Lie group actions. For our given setup, the following hold for every $x \in A \cap U$.

- (3.1) The linear map $\Omega_x : \wedge^2 T_x \mathcal{O}^\perp \rightarrow \mathfrak{g}_{2(2)}$ intertwines the homomorphism of Lie algebras $\widehat{\rho}_x : \mathfrak{g}_{2(2)} \rightarrow \mathcal{G}(x)$ for the actions of $\mathfrak{g}_{2(2)}$ on $\mathfrak{g}_{2(2)}$ and of $\mathcal{G}(x)$ on $T_x \mathcal{O}^\perp$ via λ_x^\perp .
- (3.2) The linear map $\Omega_x : \mathfrak{so}(T_x \mathcal{O}^\perp) \rightarrow \mathfrak{g}_{2(2)}$ is $\mathcal{H}_0(x)$ -invariant via λ_x^\perp . In particular, we have

$$[\lambda_x^\perp(\mathcal{H}_0(x)), \mathfrak{so}(T_x \mathcal{O}^\perp)] \subset \ker(\Omega_x).$$

Given the previous discussion there are two natural cases to consider: either $\Omega \equiv 0$, and $T\mathcal{O}^\perp$ is integrable, or for some $x \in A \cap U$ the linear map Ω_x is non-zero, and the above properties for Ω_x impose strong restrictions on the centralizer \mathcal{H} . The following result provides the description of \mathcal{H} in the latter case.

Proposition 3.5. *For a $G_{2(2)}$ -action on M as above, for A and U as in Propositions 3.1 and 3.2, respectively, let $x \in A \cap U$ be such that $\Omega_x \neq 0$. If $\mathcal{V}(x)$ is a $\mathcal{G}(x)$ -submodule of \mathcal{H} given as in Proposition 3.3, then $\mathcal{V}(x) \simeq \mathbb{R}^{3,4}$ as $\mathfrak{g}_{2(2)}$ -modules and $\dim(M) \leq 21$. Furthermore, $\mathcal{H}_0(x) = 0$ and $\mathcal{H} \simeq \mathfrak{so}(3, 4)$ both as Lie algebras and as modules over $\mathcal{G}(x) \simeq \mathfrak{g}_{2(2)}$.*

Proof. For our given x , property (3.1) implies that $T_x \mathcal{O}^\perp$ is a non-trivial $\mathfrak{g}_{2(2)}$ -module. Hence, Proposition 2.1 shows that $\mathcal{V}(x) \simeq T_x \mathcal{O}^\perp \simeq \mathbb{R}^{3,4}$ as $\mathfrak{g}_{2(2)}$ -modules. Furthermore, by Proposition 2.2 this isomorphism is an isometry up to a constant. In particular, the representation $\lambda_x^\perp : \mathcal{G}(x) \rightarrow \mathfrak{so}(T_x \mathcal{O}^\perp)$ discussed above is naturally equivalent to the linear realization $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$.

Hence, Proposition 2.5 yields the existence of a $\mathcal{G}(x)$ -module V of $\mathfrak{so}(T_x\mathcal{O}^\perp)$ such that

$$\begin{aligned}\mathfrak{so}(T_x\mathcal{O}^\perp) &= \lambda_x^\perp(\mathcal{G}(x)) \oplus V, \\ [V, V] &= \mathfrak{so}(T_x\mathcal{O}^\perp),\end{aligned}$$

for the brackets of $\mathfrak{so}(T_x\mathcal{O}^\perp)$. Thus, the map $\Omega_x : \mathfrak{so}(T_x\mathcal{O}^\perp) \rightarrow \mathfrak{g}_{2(2)}$ is naturally identified with the projection onto the summand $\lambda_x^\perp(\mathcal{G}(x))$, which implies that $\ker(\Omega_x) = V$. If we apply property (3.2) and the fact that $[V, V] = \mathfrak{so}(T_x\mathcal{O}^\perp)$, we conclude that $\lambda_x^\perp(\mathcal{H}_0(x)) = 0$. By the previous remarks in this section this yields $\mathcal{H}_0(x) = 0$.

Hence, we have $\mathcal{H} = \mathcal{G}(x) \oplus \mathcal{V}(x)$ where $\mathcal{V}(x)$ is a $\mathcal{G}(x)$ -submodule isomorphic through ev_x to $T_x\mathcal{O}^\perp \simeq \mathbb{R}^{3,4}$.

If we consider a Levi decomposition $\mathcal{H} = \mathcal{L} \oplus \text{rad}(\mathcal{H})$ such that $\mathcal{G}(x) \subset \mathcal{L}$, then such sum is a decomposition into $\mathcal{G}(x)$ -submodules as well. In particular, either $\text{rad}(\mathcal{H}) = 0$ or $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$. In the latter case, we obtain a semi-direct product $\mathcal{H} = \mathcal{G}(x) \ltimes \mathcal{V}(x)$.

Suppose that $\text{rad}(\mathcal{H}) = \mathcal{V}(x)$ and choose R a simply connected Lie group whose Lie algebra is $\mathcal{V}(x)$. Hence, the Lie group $G_{2(2)} \ltimes R$, with the semi-direct product structure, has Lie algebra \mathcal{H} . Let $\psi : \mathfrak{g}_{2(2)} \ltimes \mathcal{V}(x) \rightarrow \mathcal{H}$ be the isomorphism whose restriction to $\mathfrak{g}_{2(2)}$ is $\widehat{\rho}_x$ and that maps $\mathcal{V}(x)$ to itself by the identity. By Lemma 1.11 from [6] (or by the results from [8]), the completeness of \widetilde{M} and the fact that $\mathcal{H} \subset \text{Kill}(\widetilde{M})$ imply the existence of a right $G_{2(2)} \ltimes R$ -action on \widetilde{M} such that

$$\psi(X) = X^*,$$

for every $X \in \mathfrak{g}_{2(2)} \ltimes \mathcal{V}(x)$, where X^* denotes the Killing field generated by the (right) action of the 1-parameter subgroup $(\exp(tX))_t$ of $G_{2(2)} \ltimes R$. Consider the analytic map

$$\begin{aligned}f : G_{2(2)} \ltimes R &\rightarrow \widetilde{M} \\ f(g, r) &= x(g, r),\end{aligned}$$

given by the $G_{2(2)} \ltimes R$ -orbit at x and which is clearly $G_{2(2)} \ltimes R$ -equivariant. A straightforward computation (compare with the proof of Proposition 4.4 from [7]) shows that $df_{(e,e)}$ is an isomorphism that maps

$$df_{(e,e)}(\mathfrak{g}_{2(2)}) = T_x\mathcal{O}, \quad df_{(e,e)}(\mathcal{V}(x)) = T_x\mathcal{O}^\perp.$$

In particular, f is a local diffeomorphism from a neighborhood of the identity onto a neighborhood of x . If we choose $N = f(\{e\} \times R)$, then N is a submanifold of \widetilde{M} in a neighborhood of x such that

$$T_x N = T_x \mathcal{O}^\perp.$$

Furthermore, the equivariance of f is easily seen to imply that

$$T_{f(e,r)} N = T_{f(e,r)} \mathcal{O}^\perp,$$

for every r in a neighborhood of e in R . In other words, N is an integral submanifold of $T\mathcal{O}^\perp$ passing through x . Next, the equivariance with respect to $G_{2(2)}$ implies that there is an integral submanifold of $T\mathcal{O}^\perp$ passing through every point in a neighborhood of x . By the analyticity of $T\mathcal{O}^\perp$, we conclude that this vector bundle is integrable. This yields a contradiction since we assumed that $\Omega \neq 0$.

From the previous discussion we conclude that $\text{rad}(\mathcal{H}) = 0$ and so that \mathcal{H} is semisimple. We observe that every decomposition of \mathcal{H} into simple ideals is also a decomposition into $\mathcal{G}(x)$ -submodules. Since \mathcal{H} is the sum of the two inequivalent irreducible $\mathcal{G}(x)$ -submodules $\mathcal{G}(x)$ and $\mathcal{V}(x)$, if \mathcal{H} is not simple, then both submodules are ideals. But this is impossible because $[\mathcal{G}(x), \mathcal{V}(x)] \neq 0$. We conclude that \mathcal{H} is a simple Lie algebra of dimension 21. In particular, \mathcal{H} is a noncompact real form of the 21-dimensional simple complex Lie algebra $\mathcal{H}^{\mathbb{C}}$. An inspection of the list of simple complex Lie algebras (see [3]) shows that the only possibilities are either $\mathcal{H}^{\mathbb{C}} \simeq \mathfrak{so}(7, \mathbb{C})$ or $\mathcal{H}^{\mathbb{C}} \simeq \mathfrak{sp}(6, \mathbb{C})$. The latter and the fact that $\mathfrak{g}_{2(2)} \simeq \mathcal{G}(x) \subset \mathcal{H}$ would imply the existence of a non-trivial 6-dimensional representation of $\mathfrak{g}_2^{\mathbb{C}}$, which is absurd. We conclude that $\mathcal{H}^{\mathbb{C}} \simeq \mathfrak{so}(7, \mathbb{C})$, and so that $\mathcal{H} \simeq \mathfrak{so}(p, q)$ for some $p, q \geq 1$ such that $p + q = 7$. Considering again the inclusion $\mathfrak{g}_{2(2)} \simeq \mathcal{G}(x) \subset \mathcal{H}$ we obtain a non-trivial representation $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(p, q)$. Then, Propositions 2.1 and 2.2 imply that we must have $\{p, q\} = \{3, 4\}$ and so we can in fact assume that $p = 3, q = 4$. In other words, we conclude that $\mathcal{H} \simeq \mathfrak{so}(3, 4)$. The arguments also show that the inclusion $\mathcal{G}(x) \subset \mathcal{H}$ must correspond to the linear realization $\mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$ and so the isomorphism $\mathcal{H} \simeq \mathfrak{so}(3, 4)$ holds in the sense of $\mathcal{G}(x)$ -modules as well. \square

4. PROOF OF THE MAIN RESULTS.

In what follows we will assume that the hypotheses of Theorem 1.1 hold. We will consider two cases according to whether $T\mathcal{O}^{\perp}$ is integrable or not. In the first case, the results from [11] imply that the first conclusion from both Theorems 1.1 and 1.2 hold. So we can assume that the conclusions from Proposition 3.5 hold at some point x_0 .

Hence, the isomorphism $\widehat{\rho}_{x_0} : \mathfrak{g}_{2(2)} \rightarrow \mathcal{G}(x_0)$ can be extended to an isomorphism

$$\psi : \mathfrak{so}(3, 4) \rightarrow \mathcal{H}.$$

As in the proof of Proposition 3.5 and by the geodesic completeness of \widetilde{M} , we can apply Lemma 1.11 from [6] or the results from [8] to obtain an isometric right $\widetilde{\text{SO}}_0(3, 4)$ -action on \widetilde{M} such that

$$\psi(X) = X^*,$$

for every $X \in \mathfrak{so}(3, 4)$. Recall that X^* is the Killing field obtained from the (right) action of the 1-parameter subgroup $(\exp(tX))_t$ of $\widetilde{\text{SO}}_0(3, 4)$.

Let us denote by

$$\begin{aligned} \varphi : \widetilde{\text{SO}}_0(3, 4) &\rightarrow \widetilde{M} \\ g &\mapsto x_0 g, \end{aligned}$$

the $\widetilde{\text{SO}}_0(3, 4)$ -orbit map at x_0 . From the previous remarks it follows that

$$d\varphi_e(X) = ev_{x_0}(\psi(X))$$

for every $X \in \mathfrak{so}(3, 4)$, and so defines an isomorphism. Since φ is $\widetilde{\text{SO}}_0(3, 4)$ -equivariant, we conclude that φ is a local diffeomorphism.

Let us denote with K the Killing form of $\mathfrak{so}(3, 4)$ and let h_K be the bi-invariant pseudo-Riemannian metric on $\widetilde{\text{SO}}_0(3, 4)$ induced by K . It is well known that $\widetilde{\text{SO}}_0(3, 4)$ is complete with the pseudo-Riemannian metric h_K .

Let V be the $\mathfrak{g}_{2(2)}$ -submodule of $\mathfrak{so}(3, 4)$ complementary to $\mathfrak{g}_{2(2)}$, as given by Proposition 2.5. We have proved that $d\varphi_e = ev_{x_0} \circ \psi$ and so it defines an isomorphism of modules from $\mathfrak{so}(3, 4)$ onto $T_{x_0}\widetilde{M}$ for the module structures over $\mathfrak{g}_{2(2)}$ and $\mathcal{G}(x_0)$, respectively, and with respect to the isomorphism $\widehat{\rho}_{x_0} : \mathfrak{g}_{2(2)} \rightarrow \mathcal{G}(x_0)$. Furthermore, we also have

$$d\varphi_e(\mathfrak{g}_{2(2)}) = T_{x_0}\mathcal{O}, \quad d\varphi_e(V) = T_{x_0}\mathcal{O}^\perp.$$

On the other hand, the restrictions of the metric h_{x_0} to both $T_{x_0}\mathcal{O}$ and $T_{x_0}\mathcal{O}^\perp$ are non-degenerate and $\mathcal{G}(x_0)$ -invariant. It follows that the bilinear forms

$$\varphi_e^*(h|_{T_{x_0}\mathcal{O}}), \quad \varphi_e^*(h|_{T_{x_0}\mathcal{O}^\perp})$$

on $\mathfrak{g}_{2(2)}$ and V , respectively, are non-degenerate and $\mathfrak{g}_{2(2)}$ -invariant. By Proposition 2.2 there exists non-zero constants c_1, c_2 such that

$$K|_{\mathfrak{g}_{2(2)}} = c_1\varphi_e^*(h|_{T_{x_0}\mathcal{O}}), \quad K|_V = c_2\varphi_e^*(h|_{T_{x_0}\mathcal{O}^\perp}).$$

If we consider the pseudo-Riemannian metric on M given by

$$\overline{h} = c_1h|_{T\mathcal{O}} \oplus c_2h|_{T\mathcal{O}^\perp},$$

then the above discussion shows that the map

$$d\varphi_e : (\mathfrak{so}(3, 4), K) \rightarrow (T_{x_0}\widetilde{M}, \overline{h}_{x_0})$$

is an isometry. Furthermore, the equivariance of φ implies that it defines a local isometry $(\widetilde{\text{SO}}(3, 4), h_K) \rightarrow (\widetilde{M}, \overline{h})$. The completeness of h_K and the results from [8] prove that φ is in fact an isometry.

Let us consider the (left) $G_{2(2)}$ -action on \widetilde{M} lifted from the corresponding action on M . This yields from the isometry φ a homomorphism

$$\rho : G_{2(2)} \rightarrow \text{Iso}_0(\widetilde{\text{SO}}_0(3, 4), h_K).$$

The latter group of isometries is given by $L(\widetilde{\text{SO}}_0(3, 4))R(\widetilde{\text{SO}}_0(3, 4))$, the group of left and right translations of $\widetilde{\text{SO}}_0(3, 4)$, and so we obtain a pair of homomorphisms

$$\rho_1, \rho_2 : G_{2(2)} \rightarrow \widetilde{\text{SO}}(3, 4),$$

such that

$$\rho(g) = L_{\rho_1(g)}R_{\rho_2(g)^{-1}},$$

for every $g \in G_{2(2)}$. We note that this $G_{2(2)}$ -action commutes with the right $\widetilde{\text{SO}}_0(3, 4)$ -action on \widetilde{M} and so both actions commute when acting on $\widetilde{\text{SO}}_0(3, 4)$. This implies that

$$\rho_2(G_{2(2)}) \subset Z(\widetilde{\text{SO}}_0(3, 4)),$$

thus showing that $\rho_2 = e$. In particular, we have $\rho = L_{\rho_1}$, i.e. the $G_{2(2)}$ -action defined by ρ is given by left translations by ρ_1 . Hence, φ is $G_{2(2)}$ -equivariant for the $G_{2(2)}$ -action on the domain given by the non-trivial homomorphism $\rho_1 : G_{2(2)} \rightarrow \widetilde{\text{SO}}_0(3, 4)$ and left translations.

Let us identify \widetilde{M} with $\widetilde{\text{SO}}_0(3, 4)$ through the isometry φ . By the previous discussion we have

$$\pi_1(M) \subset \text{Iso}(\widetilde{\text{SO}}_0(3, 4), h_K).$$

Since $\text{Iso}_0(\widetilde{\text{SO}}_0(3, 4))$ has finite index in $\text{Iso}(\widetilde{\text{SO}}_0(3, 4))$ (see for example [10]) we conclude that the discrete subgroup

$$\Gamma_1 = \pi_1(M) \cap \text{Iso}_0(\widetilde{\text{SO}}_0(3, 4), h_K) = \pi_1(M) \cap L(\widetilde{\text{SO}}_0(3, 4))R(\widetilde{\text{SO}}_0(3, 4))$$

is a finite index subgroup of $\pi_1(M)$. Every element $\gamma \in \Gamma_1$ corresponds to an isometry

$$\gamma = L_{g_1} R_{g_2},$$

where $g_1, g_2 \in \widetilde{\mathrm{SO}}_0(3, 4)$. Since the Γ_1 -action and the lifted $G_{2(2)}$ -action on \widetilde{M} commute with each other, it follows that $g_1 \in Z = Z_{\widetilde{\mathrm{SO}}_0(3, 4)}(\rho_1(G_{2(2)}))$, the centralizer in $\widetilde{\mathrm{SO}}_0(3, 4)$ of the image of $\rho_1 : G_{2(2)} \rightarrow \widetilde{\mathrm{SO}}_0(3, 4)$. Hence, we conclude that

$$\Gamma_1 \subset L(Z)R(\widetilde{\mathrm{SO}}_0(3, 4)).$$

We now prove the following.

Lemma 4.1. *For every non-trivial homomorphism $\rho_1 : G_{2(2)} \rightarrow \widetilde{\mathrm{SO}}_0(3, 4)$, the centralizer $Z = Z_{\widetilde{\mathrm{SO}}_0(3, 4)}(\rho_1(G_{2(2)}))$ of the image of ρ_1 in $\widetilde{\mathrm{SO}}_0(3, 4)$ is a finite subgroup.*

Proof. Consider the corresponding non-trivial homomorphism $d\rho_1 : \mathfrak{g}_{2(2)} \rightarrow \mathfrak{so}(3, 4)$, and let V a $\mathfrak{g}_{2(2)}$ -submodule of $\mathfrak{so}(3, 4)$ complementary to $\mathfrak{g}_{2(2)}$. By Proposition 2.1 we have $V \simeq \mathbb{R}^{3, 4}$ as $\mathfrak{g}_{2(2)}$ -modules. It follows that $d\rho_1(\mathfrak{g}_{2(2)})$ is a maximal subalgebra of $\mathfrak{so}(3, 4)$. Since $d\rho_1(\mathfrak{g}_{2(2)}) + \mathfrak{z}$ is a Lie subalgebra of $\mathfrak{so}(3, 4)$, where \mathfrak{z} is the Lie algebra of Z , we conclude that $\mathfrak{z} = 0$. Hence, Z is a discrete subgroup. Finally, Lemma 1.1.3.7 from [14] implies that Z is contained in any maximal compact subgroup of $\widetilde{\mathrm{SO}}_0(3, 4)$. Hence, Z is a finite subgroup. \square

By Lemma 4.1 we conclude that

$$\Gamma = \Gamma_1 \cap R(\widetilde{\mathrm{SO}}_0(3, 4)),$$

is a finite index subgroup of Γ_1 and so of $\pi_1(M)$. The group Γ is clearly identified with a discrete subgroup of $\widetilde{\mathrm{SO}}_0(3, 4)$ such that

$$\pi : \widehat{M} = \widetilde{\mathrm{SO}}_0(3, 4)/\Gamma \rightarrow \widetilde{M}/\pi_1(M) = M$$

defines a finite cover of M .

On the other hand, a proof similar to that of Lemma 3.4 from [6] shows that M has finite volume on the metric \bar{h} . Hence, Γ is a lattice of $\widetilde{\mathrm{SO}}_0(3, 4)$. This proves that the cases (2) of Theorems 1.1 and 1.2 are satisfied, thus completing the proof of these theorems.

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